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The disc amplitude of the Dijkgraaf-Vafa theory: $1/N$ expansion vs. complex curve analysis

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Abstract: According to Dijkgraaf and Vafa the effective glueball superpotential of the $\mathcal{N}=1$ supersymmetric QCD coupled with an adjoint chiral multiplet is given by the planar amplitude in the $1/N$ expansion of a matrix model. It was shown that, when the $\mathcal{N}=1$ supersymmetric QCD is coupled with fundamental chiral multiplets as well, the effective glueball superpotential is modified by the disc amplitude of the generalized matrix model. The diagramatic computation of this disc amplitude is fairly involved for the multi-cut solution. Instead we compute it with recourse to the complex analysis of the hyperelliptic curve. The result is given in series of the gluino condensation S_i. The explicit computation for the generic multi-cut solution is done up to order S^3. It is systematic so that it can be extended to higher orders.

Keywords: Matrix Models, Nonperturbative Effects, Supersymmetric Effective Theories, 1/N Expansion.

Contents

1. Introduction

Dijkgraaf and Vafa discovered a close relation between the $\mathcal{N} = 1$ supersymmetric QCD and a matrix model. They claimed that the glueball superpotential of the $\mathcal{N} = 1$ supersymmetric QCD coupled with an adjoint chiral multiplet can be computed by the $\frac{1}{N}$ expansion of an associated matrix model [1, 2]. It was confirmed by subsequent works by many people. The effective theory of the $\mathcal{N} = 1$ supersymmetric QCD given by such a glueball superptential is called the Dijkgraaf-Vafa (DV) theory. In [1] they developed a remarkable technique to compute the planar diagram amplitude of the matrix model. Namely the computation, which was normally done by the perturbative sum of the planar diagrams, was equivalently replaced by the analysis of the hyperelliptic curve on the Riemann surface. This revealed important aspects underlying in the $\mathcal{N} = 1$ supersymmetric QCD like the duality [3], the integrable hierarchy, *etc*, and attracted a renewed attention [4−6] to the Seiberg-Witten theory for the $N = 2$ supersymmetric QCD [7].

In [8−11] the relation between the the $\mathcal{N} = 1$ supersymmetric QCD and a matrix model was generalized for the case in which the $\mathcal{N} = 1$ supersymmetric QCD is coupled with fundamental chiral multiplets as well. The presence of fundamental chiral multiplets modifies the glueball superpotential of the $\mathcal{N} = 1$ supersymmetric QCD. It was shown that the modification can be captured by computing the planar diagrams with boundary (discs) of the matrix model which is generalized accordingly. The disc amplitude is the perturbative sum of such disc diagrams. It was computed to all orders of the gluino condensation S for the 1-cut solution [8], but merely to order S^2 for the n-cut solution [10]. The computation to higher orders for the n-cut solution is extremely involved. In this paper we will discuss that the disc amplitude for the n-cut solution may be computed to higher orders in a rather systematic way, if one has recourse to the complex analysis of the hyperelliptic curve. Indeed the planar amplitude for the 2-cut solution was calculated along this line to order S^3 in [12]. This calculation to order S^3 was generalized for the n-cut solution in [13] later on. Our aim of this paper is to generalize the calculation of the planar amplitude in [13] to that of the disc amplitude.

The paper is organized as follows. We give a short review on the DV theory coupled with fundamental chiral multiplets in section 2. The disc amplitude of the associated matrix model is given in terms of the (hyper)elliptic function. As a warming-up it is evaluated for the 1-cut solution in section 3. The evaluation is exact for this case. For the multi-cut solution it can be done only in series of the gluino condensation S. In section 4 the disc amplitude for the 2-cut solution is evaluated to order S^3. In section 5 this evaluation is generalized for the n-cut solution. Section 6 is devoted to shed light on the geometric structure of the DV theory which is obscure in our analysis.

2. A short review on the DV theory

We consider the $\mathcal{N}=1$ supersymmetric $\mathrm{U}(N)$ gauge theory with the superpotential

$$W(\Phi, Q^I, \tilde{Q}_I) = \operatorname{tr} V(\Phi) + \sum_{I=1}^{N_f} (m_I \tilde{Q}_I Q^I + Q_I \Phi Q^I)\,, \tag{2.1}$$

Here Φ is an adjoint chiral multiplet and $Q^I(\tilde{Q}_I)$ are N_f chiral multiplets in the fundamental representation. $V(\Phi)$ is a polynomial scalar potential of degree M_0 given by

$$V'(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{M_0})\,. \tag{2.2}$$

When the $\mathrm{U}(N)$ gauge group symmetry is broken to $\Pi_i\,\mathrm{U}(N_i)$ with $\sum_{i=1}^n N_i = N$, this gauge theory is described by the effective action

$$W_{\mathrm{eff}} = \sum_{i=1}^{n} N_i \frac{\partial F_2(S)}{\partial S_i} + F_1(S) + 2\pi i \tau_0 \sum_{i=1}^{n} S_i\,, \tag{2.3}$$

for the gluino condensation S_i. It was claimed that $F_2(S)$ and $F_1(S)$ are the planar and disc amplitudes in the $\frac{1}{M}$ expansion of the partition function of the associated matrix model

$$\begin{aligned} Z &= \int d\phi\, dq^I\, d\tilde{q}_I\, \exp\left[-\frac{1}{g_s} W(\phi, q^I, \tilde{q}_I)\right] \\ &= \exp\left[-\sum_{\chi\le 2} g_s^{-\chi} F_\chi(S)\right]. \end{aligned} \tag{2.4}$$

Here ϕ is a $M\times M$ matrix and $q^I(\tilde{q}^I)$ are M-dimensional vectors. W is the superpotential given by (2.1) so that the matrix model is gauge-invariant under $\mathrm{U}(M)$. The gluino condensation S_i is identified with the 'tHooft coupling $g_s M_i$, when the gauge symmetry $\mathrm{U}(M)$ is broken to $\Pi_i\,\mathrm{U}(M_i)$ with $\sum_{i=1}^n M_i = M$. The planar and disc amplitudes, respectively

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given by F_2 and F_1, are calculated by an alternative method. Namely in the limit $M \to \infty$ we obtain the saddle point equation for the eigenvalues of ϕ distributed with density $\rho(\lambda)$

$$y \equiv V'(x) - 2\int d\lambda \frac{\varrho(\lambda)}{x-\lambda} + \sum_{I=1}^{N_f} \frac{g_s}{x+m_I} = 0 \tag{2.5}$$

with the normalization $\int d\lambda\, \rho(\lambda) = S(= g_s M)$. Then the partition function (2.4) may be written as

$$Z = \exp\Biggl(-\frac{M^2}{S^2}\biggl[\int d\lambda\, \rho(\lambda)V(\lambda) - \int\int d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda')\log(\lambda-\lambda')\biggr] - \\ - \frac{M}{S}\sum_{I=1}^{N_f}\int d\lambda\, \rho(\lambda)\log(\lambda+m_I) + O\biggl(\Bigl(\frac{M}{S}\Bigr)^0\biggr)\Biggr). \tag{2.6}$$

In (2.5) y is the quantum deformed force of the matrix model. For the n-cut solution with the eigenvalues spreading out along n real lines it may be given by a hyperelliptic function such that

$$y = (x-\mu_1)(x-\mu_1)\cdots(x-\mu_{M_0-n})\sqrt{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_{2n})}\,. \tag{2.7}$$

Comparing the polynomilal part in the expansion of (2.5) and (2.7) at $x = \infty$ gives the zero-points of $V'(x)$ as functions of μ's and λ's:

$$\alpha_a = \alpha_a(\mu_1, \ldots, \mu_{M_0-n}; \lambda_1 \ldots, \lambda_{2n})\,, \qquad a = 1, 2, \ldots, M_0\,. \tag{2.8}$$

The hyperelliptic function (2.7) describes a Riemann surface Σ with genus $n-1$ as a double cover of the complex z-plane with cuts along the real lines $[\lambda_{2i}, \lambda_{2i+1}]$, $i = 1, \ldots, n$. Let $P(\tilde{P})$ be the point on the first(second) sheet at $z = \infty$. Denote by C_i a path from $\tilde{P}$ to P which passes through the ith cut $[\lambda_{2i}, \lambda_{2i+1}]$. The canonical homology basis of the Riemann surface Σ is given by the cicle A_i surrounding the ith cut and the dual one defined $B_i = C_i - C_n$. We consider the DV differential ydx defined with such a hyperelliptic function. For the n-cut solution the gluino condensation is given by

$$S_i = \frac{1}{4\pi i}\oint_{A_i} ydx\,, \qquad i = 1, 2, \ldots, n\,. \tag{2.9}$$

This gives S_i as functions of μ's and λ's:

$$S_i = S_i(\mu_1, \ldots, \mu_{M_0-n}; \lambda_1 \ldots, \lambda_{2n})\,. \tag{2.10}$$

We solve (2.8) and (2.10) for μ's and λ's to find them as functions of α's and S's:

$$\begin{aligned} &\mu_\kappa(S_1, \ldots, S_n; \alpha_1, \ldots, \alpha_{M_0}), \qquad && \kappa = 1, 2, \ldots, M_0 - n\,, \\ &\lambda_r(S_1, \ldots, S_n; \alpha_1, \ldots, \alpha_{M_0}), \qquad && r = 1, 2, \ldots, 2n\,. \end{aligned} \tag{2.11}$$

From (2.6) the planar amplitude reads

$$F_2 = \int d\lambda\, \rho(\lambda)V(\lambda) - \int\int d\lambda\, d\lambda'\, \rho(\lambda)\rho(\lambda')\log(\lambda-\lambda')\,, \tag{2.12}$$

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which satisfies

$$\frac{\partial F_2}{\partial S_i} = \frac{1}{2}\int_{B_i} ydx + \frac{1}{2}\int_{C_N} ydx. \tag{2.13}$$

On the other hand the disc amplitude reads

$$\begin{aligned} F_1 &= \sum_{I=1}^{N_f} \int d\lambda\, \rho(\lambda) \int_{-m_I}^{\Lambda} \frac{dx}{\lambda - x} \\ &= \frac{1}{2}\sum_{I=1}^{N_f} \int_{-m_I}^{\Lambda} dx[y - V'(x)]\,, \end{aligned} \tag{2.14}$$

with a cut-off parameter Λ at P.

We want to calculate this disc amplitude for the n-cut solution and compare it with the result obtained by the $\frac{1}{M}$ expansion. For simplicity we shall do it simply assuming that $M_0 = n$.

3. The 1-cut solution

As a warming-up we start with the simplest example, i.e., the matrix model with the scalar potential

$$V(x) = \frac{1}{2}\xi x^2\,,$$

which is coupled with one chiral matter multiplet. We consider the DV differential

$$ydx = \sqrt{V'(x)^2 - 4\xi S}dx\,,$$

on the Riemann surface with one branch-cut. Note that the branch points are chosen so that (2.10) is satisfied, i.e.,

$$S = \frac{1}{2\pi i}\int_{\sqrt{-4S/\xi}}^{\sqrt{4S/\xi}} y\, dx\,.$$

The disc amplitude (2.14) is exactly calculated as

$$F_1(S) = -\frac{S}{2} - \frac{1}{4\alpha}(\sqrt{1-4\alpha S} - 1) + S\log\frac{1+\sqrt{1-4\alpha S}}{2} + S\log\frac{m}{\Lambda}\,, \tag{3.1}$$

with $\alpha = \frac{1}{m^2\xi}$. This is in complete agreement with the result obtained by the $\frac{1}{N}$ expansion [8]. The log-term which is linear in S is a non-perturbative part and may be absorbed by the last term in the effective action (2.3). A similar calculation for the 1-cut solution was done for a variant of the matrix model in [14].

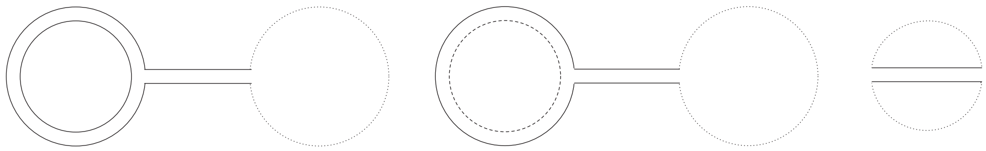

Figure 1: The Feynman diagrams to order S^2 corresponding to the three terms in (4.2). The solid double line and the single dashed line are propagators for ϕ and q^I respectively. The solid and dashed double line is a propagator for ghosts.

4. The 2-cut solution

Next we study the matrix model in which the scalar potential is given by

$$V'(x) = (x - \alpha_1)(x - \alpha_2)\,. \tag{4.1}$$

In [10] they calculated Feynmann diagrams dipicted in figure 1 to find the disc amplitude to order S^2:

$$F_1(S) = -\sum_{I=1}^{N_f}\sum_{i=1}^{2}\left[\frac{S_i^2}{\Delta_i(\alpha) f_{iI}}\frac{1}{\alpha_{ij}} - 2\sum_{j\neq i}^{2}\frac{S_i S_j}{\Delta_i(\alpha)\alpha_{ij} f_{iI}} + \frac{1}{2}\frac{S_i^2}{\Delta_i(\alpha) f_{iI}^2}\right]. \tag{4.2}$$

Here $\Delta_i(\alpha) = \prod_{j\neq i}^{2}\alpha_{ij}$, $f_{iI} = m_I + \alpha_i$, $\alpha_{ij} = \alpha_i - \alpha_j$. In this section we shall evaluate the formula (2.14) for the 2-cut solution to order S^3 and show that it indeed reproduces the result (4.2) to order S^2. To this end we consider the DV differential

$$y dx = \sqrt{\left((x-\gamma_1)^2 - \rho_1^2\right)\left((x-\gamma_2)^2 - \rho_2^2\right)}\, dx\,, \tag{4.3}$$

on the Riemann surface with two branch-cuts $[\gamma_1 - \alpha_1, \gamma_1 + \alpha_1]$ and $[\gamma_2 - \alpha_2, \gamma_2 + \alpha_2]$. Here y may be written in the form (2.5). Hence comparing the polynomial part in the expansion of (2.5) and (4.3) at $x = \infty$ yields the relations $\alpha_i(\rho^2, \gamma)$:

$$\alpha_1 + \alpha_2 = \gamma_1 + \gamma_2\,, \tag{4.4}$$

$$(\alpha_1 + \alpha_2)^2 + 2\alpha_1\alpha_2 = \gamma_1^2 + \gamma_2^2 + 4\gamma_1\gamma_2 - \rho_1^2 - \rho_2^2\,. \tag{4.5}$$

With the DV differential (4.3) we will calculate the disc amplitude (2.14). A suitable technique to this end was developed for the calculation of the planar amplitude in [13]. The calculation goes following the steps:

1. evaluation of the period integrals

$$S_i = \frac{1}{2\pi i}\int_{\gamma_i - \rho_i}^{\gamma_i + \rho_i} y\, dx\,, \tag{4.6}$$

which provides explicit formulae $S_i(\rho^2, \gamma)$ for gluino condensation;

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2. inversion of the formula $S_i(\rho^2,\gamma)$ which provides $\rho_i^2(S,\gamma)$ in series of S_i with coefficients made out of γ_i;

3. inversion of the formulae $\alpha_i(\rho^2,\gamma)$, given by (4.4) and (4.5), which provides $\gamma_i(\rho^2,\alpha)$ in series of ρ_i^2 with coefficients made out of α_i;

4. evaluation of the DV differential (4.3) in series of ρ_i^2 with coefficients made out of γ_i;

5. expansion of ρ_i^2 and γ_i in the above result in series of S_i with the coefficients made out of α_i by iterating the inversion formulae obtained at Step 2 and 3.

Step 1. After the shift $x' = x - \gamma_1$ we expand the DV differential (4.3) in series of ρ_2^2 as

$$ydx = \sum_{m=0}^{\infty} c_m \rho_2^{2m} (x' + \gamma_{12})^{1-2m} \sqrt{x'^2 - \rho_1^2}\, dx' \,, \tag{4.7}$$

where $\gamma_{12} = \gamma_1 - \gamma_2$ and $c_m = \frac{\Gamma(m-\frac{1}{2})}{m!\Gamma(-\frac{1}{2})}$. Write $(x'+\gamma_{12})^{1-2m}$ for $m \geq 1$ as $(2m-2)$-ple derivative of $(x'+\gamma_{12})^{-1}$ by γ_{12}. Then it takes the form

$$ydx = \left[(x'+\gamma_{12})\sqrt{x'^2-\rho_1^2} + \sum_{m\geq 1} \frac{c_m}{(2m-2)!}\rho_2^{2m}\left(\frac{\partial}{\partial\gamma_{12}}\right)^{2m-2} \frac{\sqrt{x'^2-\rho_1^2}}{x'+\gamma_{12}}\right] dx' \,. \tag{4.8}$$

We integrate the DV differential of this form along the A_1-cycle surrounding the branch-cut $[\gamma_1 - \rho_1, \gamma_1 + \rho_1]$. We find

$$i\int_{-\rho_1}^{\rho_1} dx'\, (x'+\gamma_{12})\sqrt{\rho_1^2 - x'^2} = \frac{i\pi}{2}\gamma_{12}\rho_1^2 \,, \tag{4.9}$$

for the first piece and

$$i\int_{-\rho_1}^{\rho_1} dx'\, \frac{\sqrt{\rho_1^2 - x'^2}}{x'+\gamma_{12}} = i\pi\left(-\sqrt{\gamma_{12}^2 - \rho_1^2} + \gamma_{12}\right) , \tag{4.10}$$

for the second piece. Putting them together in (4.6) then we get the expression for the gluino condensation S_1

$$S_1 = \frac{1}{4}\gamma_{12}\rho_1^2 - \frac{\rho_1^2\rho_2^2}{8\gamma_{12}} - \frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{32\gamma_{12}^3} + O(\rho^8) \,. \tag{4.11}$$

Similarly we get for the gluino condensation S_2

$$S_2 = -\frac{1}{4}\gamma_{12}\rho_2^2 + \frac{\rho_1^2\rho_2^2}{8\gamma_{12}} + \frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{32\gamma_{12}^3} + O(\rho^8) \,. \tag{4.12}$$

Step 2. Inverting the relations (4.11) and (4.12) we obtain

$$\rho_1^2 = \frac{4S_1}{\gamma_{12}} - \frac{8S_1S_2}{\gamma_{12}^4} - \frac{24S_1S_2(S_1-S_2)}{\gamma_{12}^7} + O(S^4) \,, \tag{4.13}$$

$$\rho_2^2 = -\frac{4S_2}{\gamma_{12}} - \frac{8S_1S_2}{\gamma_{12}^4} - \frac{24S_1S_2(S_1-S_2)}{\gamma_{12}^7} + O(S^4) \,. \tag{4.14}$$

Step 3. Inverting the relations (4.4) and (4.5) we obtain

$$\begin{aligned}\gamma_1 &= \frac{1}{2}(\alpha_1+\alpha_2)+\frac{1}{2}\sqrt{(\alpha_1-\alpha_2)^2-2(\rho_1^2+\rho_2^2)}\,,\\ &= \alpha_1-\frac{\rho_1^2+\rho_2^2}{2\alpha_{12}}-\frac{(\rho_1^2+\rho_2^2)^2}{4\alpha_{12}^3}+O(\rho^6), \end{aligned}\tag{4.15}$$

$$\begin{aligned}\gamma_2 &= \frac{1}{2}(\alpha_1+\alpha_2)-\frac{1}{2}\sqrt{(\alpha_1-\alpha_2)^2-2(\rho_1^2+\rho_2^2)}\\ &= \alpha_2+\frac{\rho_1^2+\rho_2^2}{2\alpha_{12}}+\frac{(\rho_1^2+\rho_2^2)^2}{4\alpha_{12}^3}+O(\rho^6)\,. \end{aligned}\tag{4.16}$$

Step 4. We expand the DV differential (4.8) in series of ρ_1^2 as well.

$$\begin{aligned} ydx = \Bigg[&V'(x)-\frac{\gamma_{12}}{2}\left(\frac{\rho_1^2}{x'}-\frac{\rho_2^2}{x'+\gamma_{12}}\right)-\frac{1}{8}\left\{\frac{\rho_1^4}{x'^2}+\frac{\rho_2^4}{(x'+\gamma_{12})^2}\right\}-\\ &-\frac{\gamma_{12}}{8}\left\{\frac{\rho_1^4}{x'^3}-\frac{\rho_2^4}{(x'+\gamma_{12})^3}\right\}+\frac{\rho_1^2\rho_2^2}{4\gamma_{12}}\left(\frac{1}{x'}-\frac{1}{x'+\gamma_{12}}\right)-\\ &-\frac{1}{16}\left\{\frac{\rho_1^6}{x'^4}+\frac{\rho_2^6}{(x'+\gamma_{12})^4}\right\}-\frac{\gamma_{12}}{16}\left\{\frac{\rho_1^6}{x'^5}-\frac{\rho_2^6}{(x'+\gamma_{12})^5}\right\}+\\ &+\frac{1}{16\gamma_{12}}\left\{\frac{\rho_1^4\rho_2^2}{x'^3}-\frac{\rho_1^2\rho_2^4}{(x'+\gamma_{12})^3}\right\}-\frac{1}{16\gamma_{12}^2}\left\{\frac{\rho_1^4\rho_2^2}{x'^2}+\frac{\rho_1^2\rho_2^4}{(x'+\gamma_{12})^2}\right\}+\\ &+\frac{\rho_1^2\rho_2^2(\rho_1^2+\rho_2^2)}{16\gamma_{12}^3}\left(\frac{1}{x'}-\frac{1}{x'+\gamma_{12}}\right)+O(\rho^8)\Bigg]dx'\,. \end{aligned}\tag{4.17}$$

Here $V'(x)$ is given by

$$V'(x)=x'^2+\gamma_{12}x'-\frac{1}{2}(\rho_1^2+\rho_2^2)\,,$$

which is nothing but the potential (4.1) owing to (4.4) and (4.5). After shifting the variable x' back to x we write (4.17) expressing ρ_i^2 and γ_i in terms S_i and α_i with recourse to (4.13)–(4.16). In particular the terms of order ρ^2 and ρ^4 in (4.17) read

$$\begin{aligned} \frac{\gamma_{12}}{2}\left(\frac{\rho_1^2}{x'}-\frac{\rho_2^2}{x'+\gamma_{12}}\right) = \;&2\left\{S_1-\frac{2}{\alpha_{12}^3}S_1S_2-\frac{30}{\alpha_{12}^6}S_1S_2(S_1-S_2)\right\}\frac{1}{x-\alpha_1}+\\ &+2\left\{S_2+\frac{2}{\alpha_{12}^3}S_1S_2+\frac{30}{\alpha_{12}^6}S_1S_2(S_1-S_2)\right\}\frac{1}{x-\alpha_2}-\\ &-4\left\{\frac{1}{\alpha_{12}^2}S_1(S_1-S_2)+\frac{2}{\alpha_{12}^5}S_1(3S_1^2-9S_1S_2+4S_2^2)\right\}\frac{1}{(x-\alpha_1)^2}+\\ &+4\left\{\frac{1}{\alpha_{12}^2}S_2(S_1-S_2)+\frac{2}{\alpha_{12}^5}S_2(4S_1^2-9S_1S_2+3S_2^2)\right\}\frac{1}{(x-\alpha_2)^2}+\\ &+\frac{8}{\alpha_{12}^4}\frac{S_1(S_1-S_2)^2}{(x-\alpha_1)^3}+\frac{8}{\alpha_{12}^4}\frac{S_2(S_1-S_2)^2}{(x-\alpha_2)^3}\,, \end{aligned}$$

$$
\begin{aligned}
\frac{1}{8}\left\{\frac{\rho_1^4}{x'^2}+\frac{\rho_2^4}{(x'+\gamma_{12})^2}\right\} &= 2\left\{\frac{S_1^2}{\alpha_{12}^2}+\frac{4}{\alpha_{12}^5}S_1(2S_1^2-3S_1S_2)\right\}\frac{1}{(x-\alpha_1)^2}+\\
&\quad +2\left\{\frac{S_2^2}{\alpha_{12}^2}+\frac{4}{\alpha_{12}^5}S_2(3S_1S_2-2S_2^2)\right\}\frac{1}{(x-\alpha_2)^2}-\\
&\quad -\frac{8}{\alpha_{12}^4}\frac{S_1^2(S_1-S_2)}{(x-\alpha_1)^3}+\frac{8}{\alpha_{12}^4}\frac{S_2(S_1-S_2)}{(x-\alpha_2)^3}\,,\\
\frac{\gamma_{12}}{8}\left\{\frac{\rho_1^4}{x'^3}-\frac{\rho_2^4}{(x'+\gamma_{12})^3}\right\} &= 2\left\{\frac{S_1^2}{\alpha_{12}}+\frac{4}{\alpha_{12}^4}S_1^2(S_1-2S_2)\right\}\frac{1}{(x-\alpha_1)^3}-\\
&\quad -2\left\{\frac{S_2^2}{\alpha_{12}}+\frac{4}{\alpha_{12}^4}S_2^2(2S_1-S_2)\right\}\frac{1}{(x-\alpha_2)^3}-\\
&\quad -\frac{12}{\alpha_{12}^3}\frac{S_1^2(S_1-S_2)}{(x-\alpha_1)^4}-\frac{12}{\alpha_{12}^3}\frac{S_2^2(S_1-S_2)}{(x-\alpha_2)^4}\,,\\
\frac{\rho_1^2\rho_2^2}{4\gamma_{12}}\left(\frac{1}{x'}-\frac{1}{x'+\gamma_{12}}\right) &= -4\left\{\frac{S_1S_2}{\alpha_{12}^3}+\frac{14}{\alpha_{12}^6}S_1S_2(S_1-S_2)\right\}\left(\frac{1}{x-\alpha_1}-\frac{1}{x-\alpha_2}\right)+\\
&\quad +\frac{8}{\alpha_{12}^5}S_1S_2(S_1-S_2)\left\{\frac{1}{(x-\alpha_1)^2}+\frac{1}{(x-\alpha_2)^2}\right\}.
\end{aligned}
$$

With these the disc amplitude (2.14) is evaluated as

$$
\begin{aligned}
F_1(S) = -\sum_{I=1}^{N_f}\Bigg[&-S_1\log\frac{m_I+\alpha_1}{\Lambda}-S_2\log\frac{m_I+\alpha_2}{\Lambda}+\\
&+\left(\frac{1}{\alpha_{12}^2}(S_1^2-2S_1S_2)+\frac{2}{\alpha_{12}^5}(2S_1^3-9S_1^2S_2+6S_1S_2^2)\right)\frac{1}{m_I+\alpha_1}+\\
&+\left(\frac{1}{\alpha_{12}^2}(S_2^2-2S_1S_2)-\frac{2}{\alpha_{12}^5}(2S_2^3-9S_1^2S_2+6S_1^2S_2)\right)\frac{1}{m_I+\alpha_2}+\\
&+\left(\frac{S_1^2}{2\alpha_{12}}+\frac{1}{\alpha_{12}^4}(2S_1^3-5S_1^2S_2+2S_1S_2^2)\right)\frac{1}{(m_I+\alpha_1)^2}+\\
&+\left(-\frac{S_1^2}{2\alpha_{12}}+\frac{1}{\alpha_{12}^4}(2S_2^3-5S_1S_2^2+2S_1^2S_2)\right)\frac{1}{(m_I+\alpha_2)^2}+\\
&+\frac{2}{3\alpha_{12}^3}\left\{(2S_1^3-3S_1^2S_2)\frac{1}{(m_I+\alpha_1)^3}-(2S_2^3-3S_1S_2^2)\frac{1}{(m_I+\alpha_2)^3}\right\}+\\
&+\frac{1}{2\alpha_{12}^2}\left\{\frac{S_1^3}{(m_I+\alpha_1)^4}+\frac{S_2^3}{(m_I+\alpha_2)^4}\right\}+O(S^4)\Bigg]\,. \qquad (4.18)
\end{aligned}
$$

The log-terms linear in S is a non-perturbative part which may be absorbed by the last term in the effective action (2.3). The terms of order S^2 are in complete agreement with the result (4.2) by the $\frac{1}{N}$ expansion.

5. The n-cut solution

We shall generalize the calculation for the 2-cut solution to that of the n-cut solution. That is, taking the matrix model in which the scalar potential is given by

$$V'(x)=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)\,. \qquad (5.1)$$

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we consider the DV differential

$$y dx = \sqrt{\prod_{i=1}^{n} \left((x-\gamma_i)^2 - \rho_i^2\right)} dx\,, \tag{5.2}$$

on the Riemann surface with n branch-cuts. Again we calculate the disc amplitude following Step 1 to Step 5.

Step 1 and Step 2. The inversion formula $\rho_i^2(S,\gamma)$ was given in the expansion form [13]

$$\rho_i^2 = \frac{4S_i}{\Delta_i(\gamma)}\left(1 - \frac{1}{2\Delta_i(\gamma)}\sum_{\substack{j\neq k\\ j,k\neq i}} \frac{S_i}{\gamma_{ij}\gamma_{ik}} + 2\sum_{j\neq i}\frac{S_j}{\gamma_{ij}^2\Delta_j(\gamma)}\right) + O(S^3)\,, \tag{5.3}$$

in which

$$\Delta_i(\gamma) = \prod_{i\neq j}^{n} \gamma_{ij}\,, \qquad \gamma_{ij} = \gamma_i - \gamma_j\,.$$

As it will be clear, the expansion up to order S^2 is sufficient for the calculation of the disc amplitude to order S^3.

Step 3. We expand y, given by (2.5) and (5.2), at $x=\infty$. Comparing the polynomial part of both expansions yields [13]

$$2\sum_{i=1}^{n}\alpha_i^l = \sum_{i=1}^{n}\Big[(\gamma_i+\rho_i)^l + (\gamma_i-\rho_i)^l\Big], \qquad 1\leq l\leq n\,. \tag{5.4}$$

We shall express γ_i in terms of ρ and α by solving this set of equations recursively. First of all expand (5.4) in series of ρ^2 as

$$\sum_{i=1}^{n}\alpha_i^l = \sum_{i=1}^{n}\left[\gamma_i^l + \binom{l}{2}\gamma_i^{l-2}\rho_i^2 + \binom{l}{4}\gamma_i^{l-4}\rho_i^4 + O(\rho^6)\right]. \tag{5.5}$$

Assume that γ_i is further expanded in the form

$$\gamma_i = \alpha_i + \sigma_i^{(2)} + \sigma_i^{(4)} + O(\rho^6)\,, \tag{5.6}$$

with $\sigma_i^{(n)}$ the term of order ρ^n. Putting this in (5.5) we get the recursion relations for $\sigma_i^{(n)}$

$$\sum_{i=1}^{n}\left[\alpha_i^{l-1}\sigma_i^{(2)} + \frac{1}{2}(l-1)\alpha_i^{l-2}\rho_i^2\right] = 0\,, \tag{5.7}$$

$$\begin{aligned}\sum_{i=1}^{n}\Bigg[\alpha_i^{l-1}\sigma_i^{(4)} + \frac{1}{2}(l-1)\alpha_i^{l-2}(\sigma_i^{(2)})^2 + \frac{1}{2}(l-1)(l-2)\alpha_i^{l-3}\sigma_i^{(2)}\rho_i^2 +&\\ + \frac{1}{4!}(l-1)(l-2)(l-3)\alpha_i^{l-4}\rho_i^4\Bigg] = 0\,,&\end{aligned} \tag{5.8}$$

and so on. Note that the identity

$$\frac{P^{[i]}(\alpha_j)}{\Delta_i(\alpha)} = \delta_{ij} \,, \tag{5.9}$$

for the quantity

$$P^{[i]}(x) = \frac{V'(x)}{x-\alpha_i} = \prod_{l\neq i}^{n}(x-\alpha_l) \,. \tag{5.10}$$

Using the expansion

$$P^{[i]}(x) = \sum_{l=1}^{n} c_l^{[i]}(\alpha)x^{l-1} \,,$$

(5.9) becomes

$$\frac{1}{\Delta_i(\alpha)} \sum_{l=1}^{n} c_l^{[i]}(\alpha)\alpha_j^{l-1} = \delta_{ij} \,.$$

By means of this identity the recursion relations (5.7) and (5.8) can be solved for $\sigma_i^{(n)}$ as

$$\sigma_i^{(2)} = -\frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^{n} P^{[i]\prime}(\alpha_j)\rho_j^2 \,, \tag{5.11}$$

$$\begin{aligned} \sigma_i^{(4)} &= -\frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^{n} P^{[i]\prime}(\alpha_j)(\sigma_i^{(2)})^2 - \frac{1}{2\Delta_i(\alpha)} \sum_{j=1}^{n} P^{[i]\prime\prime}(\alpha_j)\sigma_i^{(2)}\rho_j^2 - \\ &\quad -\frac{1}{24\Delta_i(\alpha)} \sum_{j=1}^{n} P^{[i]\prime\prime\prime}(\alpha_j)\rho_j^4 \,. \end{aligned} \tag{5.12}$$

Here $P^{[i]\prime}(x), P^{[i]\prime\prime}(x), P^{[i]\prime\prime\prime}(x)$ are derivatives of the quantity (5.10) with respect to x. Putting (5.11) and (5.12) in (5.6) we obtain the formula generalizing (4.15) and (4.16) for the n-cut solution.

Step 4. Expanding y given by the DV differential (5.2) in series of ρ^2 we get

$$\begin{aligned} y &= -\sum_{i=1}^{n}\left[\frac{1}{2}\frac{\rho_i^2}{x-\gamma_i} + \frac{1}{8}\frac{\rho_i^4}{(x-\gamma_i)^3} + \frac{1}{16}\frac{\rho_i^6}{(x-\gamma_i)^5}\right]\tilde{P}^{[i]}(x) + \\ &\quad + \sum_{i\neq j}^{n}\left[\frac{1}{8}\frac{\rho_i^2\rho_j^2}{(x-\gamma_i)(x-\gamma_j)} + \frac{1}{16}\frac{\rho_i^2\rho_j^4}{(x-\gamma_i)(x-\gamma_j)^3}\right]\tilde{P}^{[ij]}(x) - \\ &\quad -\frac{1}{48}\sum_{i\neq j\neq k}^{n}\frac{\rho_i^2\rho_j^2\rho_k^2}{(x-\gamma_i)(x-\gamma_j)(x-\gamma_k)}\tilde{P}^{[ijk]}(x) + O(\rho^8) \,, \end{aligned} \tag{5.13}$$

with

$$\tilde{P}^{[i]}(x) = \prod_{l\neq i}^{n}(x-\gamma_l) \,, \qquad \tilde{P}^{[ij]}(x) = \prod_{l\neq i,j}^{n}(x-\gamma_l) \,,$$

$$\tilde{P}^{[ijk]}(x) = \prod_{l\neq i,j,k}^{n}(x-\gamma_l) \,.$$

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When $\tilde{P}^{[i]}(x), \tilde{P}^{[ij]}(x)$ and $\tilde{P}^{[ijk]}(x)$ are expanded at $x = \gamma_i, \gamma_j, \gamma_k$ in Taylor series, it takes the form

$$y = V'(x) + \sum_{\nu=1}^{5} \sum_{i=1}^{n} \frac{a_{\nu i}(\rho^2, \gamma)}{(x-\gamma_i)^\nu} + O(\rho^8)\,,$$

with

$$\begin{aligned}
a_{1i}(\rho^2,\gamma) \;=\; & -\frac{\rho_i^6}{384}\tilde{P}^{[i]''''}(\gamma_i) - \frac{\rho_i^4}{16}\tilde{P}^{[i]''}(\gamma_i) - \frac{\rho_i^2}{2}\tilde{P}^{[i]}(\gamma_i) + \\
& + \sum_{j\neq i}^{n}\bigg[-\frac{\rho_i^4\rho_j^2}{16\gamma_{ij}^2}\tilde{P}^{[ij]'}(\gamma_i) + \frac{\rho_i^4\rho_j^2}{32\gamma_{ij}}\tilde{P}^{[ij]''}(\gamma_i) + \\
& \qquad + \frac{\rho_i^2\rho_j^2(\rho_i^2+\rho_j^2)}{16\gamma_{ij}^3}\tilde{P}^{[ij]}(\gamma_i) + \frac{\rho_i^2\rho_j^2}{4\gamma_{ij}}\tilde{P}^{[ij]}(\gamma_i)\bigg] - \\
& - \sum_{\substack{j\neq k\\ j,k\neq i}}^{n} \frac{\rho_i^2\rho_j^2\rho_k^2}{24}\left(\frac{1}{\gamma_{ij}\gamma_{ik}} + \frac{1}{\gamma_{ij}\gamma_{jk}}\right)\tilde{P}^{[ijk]}(\gamma_i)\,, \qquad (5.14)\\
a_{2i}(\rho^2,\gamma) \;=\; & -\frac{\rho_i^6}{96}\tilde{P}^{[i]'''}(\gamma_i) - \frac{\rho_i^4}{8}\tilde{P}^{[i]'}(\gamma_i) - \sum_{j\neq i}^{n}\frac{\rho_i^4\rho_j^2}{16\gamma_{ij}^2}\tilde{P}^{[ij]}(\gamma_i) + \sum_{j\neq i}^{n}\frac{\rho_i^4\rho_j^2}{16\gamma_{ij}}\tilde{P}^{[ij]'}(\gamma_i)\,,\\
a_{3i}(\rho^2,\gamma) \;=\; & -\frac{\rho_i^6}{32}\tilde{P}^{[i]''}(\gamma_i) - \frac{\rho_i^4}{8}\tilde{P}^{[i]}(\gamma_i) + \sum_{j\neq i}^{n}\frac{\rho_i^4\rho_j^2}{16\gamma_{ij}}\tilde{P}^{[ij]}(\gamma_i)\,,\\
a_{4i}(\rho^2,\gamma) \;=\; & -\frac{\rho_i^6}{16}\tilde{P}^{[i]'}(\gamma_i)\,,\\
a_{5i}(\rho^2,\gamma) \;=\; & -\frac{\rho_i^6}{16}\tilde{P}^{[i]}(\gamma_i)\,.
\end{aligned}$$

The polynomial part $V'(x)$ was determined by the fact that y given by (5.13) may be written in the form (2.5) as well.

Step 5. Finally we use the inversion formulae for ρ_i^2 and γ_i, given by (5.3) and (5.6) respectively, to find y in the form to order S^3

$$y = V'(x) + \sum_{\nu=1}^{5}\sum_{i=1}^{n}\frac{b_{\nu i}(S)}{(x-\alpha_i)^\nu} + O(S^4)\,, \qquad (5.15)$$

with

$$\begin{aligned}
b_{1i}(S) \;=\; & -2S_i\,,\\
b_{2i}(S) \;=\; & \sum_{j\neq i}^{n}\bigg[\frac{2S_i^2-4S_iS_j}{\Delta_i(\alpha)\alpha_{ij}} + 4\frac{S_i^3-6S_i^2S_j+3S_iS_j^2}{\Delta_i(\alpha)^2\alpha_{ij}^3} - 4\frac{S_i^3-3S_i^2S_j+3S_iS_j^2}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}^3}\bigg] + \\
& + \sum_{\substack{j\neq k\\ j,k\neq i}}^{n}\bigg[4\frac{S_i^3-8S_i^2S_j+S_iS_j^2+2S_i^2S_k+4S_iS_jS_k}{\Delta_i(\alpha)^2\alpha_{ij}^2\alpha_{ik}} - 4\frac{S_i^3-4S_i^2S_j+2S_i^2S_k}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}^2\alpha_{ik}} + \\
& \qquad + 4\frac{3S_iS_j^2-4S_i^2S_j+4S_i^2S_k-4S_iS_jS_k}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}^2\alpha_{jk}} - 4\frac{S_iS_j^2+S_iS_k^2-2S_iS_jS_k}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}\alpha_{jk}^2} +
\end{aligned}$$

$$
\begin{aligned}
&+8\frac{S_iS_j^2-2S_iS_jS_k}{\Delta_i(\alpha)\Delta_k(\alpha)\alpha_{ij}\alpha_{jk}^2}-4\frac{S_i^2S_j+S_i^2S_k-2S_iS_jS_k}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}\alpha_{ik}\alpha_{jk}}\Bigg]+\\
&+\sum_{\substack{j\neq k\neq l\\ j,k,l\neq i}}^{n}\Bigg[\frac{\frac{2}{3}S_i^3+10S_i^2S_j-4S_i^2S_k-10S_i^2S_l+4S_iS_kS_l}{\Delta_i(\alpha)^2\alpha_{ij}\alpha_{ik}\alpha_{il}}+\\
&\qquad+\frac{2S_i^2S_j-\frac{16}{3}S_iS_j^2-4S_iS_jS_k+12S_iS_jS_l-4S_iS_kS_l}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}\alpha_{jk}\alpha_{jl}}-\\
&\qquad-4\frac{S_i^2S_k-S_i^2S_l}{\Delta_i(\alpha)\Delta_k(\alpha)\alpha_{ij}\alpha_{ik}\alpha_{kl}}+8\frac{S_iS_jS_k-S_iS_jS_l}{\Delta_i(\alpha)\Delta_k(\alpha)\alpha_{ij}\alpha_{jk}\alpha_{kl}}\Bigg]+O(S^4)\,,\\
b_{3i}(S)&=-\frac{2S_i^2}{\Delta_i(\alpha)}-\sum_{j\neq i}^{n}\Bigg[\frac{4S_i(S_i-S_j)(S_i-2S_j)}{\Delta_i(\alpha)^2\alpha_{ij}^2}-\frac{4S_i^2(S_i-2S_j)}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}^2}\Bigg]-\\
&\quad-\sum_{\substack{j\neq k\\ j,k\neq i}}^{n}\Bigg[\frac{4S_i(S_i-S_j)(S_i-2S_k)}{\Delta_i(\alpha)^2\alpha_{ij}\alpha_{ik}}-\frac{4S_i^2(S_j-S_k)}{\Delta_i(\alpha)\Delta_j(\alpha)\alpha_{ij}\alpha_{jk}}\Bigg]+O(S^4)\,,\\
b_{4i}(S)&=\sum_{j\neq i}^{n}\frac{4S_i^2(2S_i-3S_j)}{\Delta_i(\alpha)^2\alpha_{ij}}+O(S^4)\,,\\
b_{5i}(S)&=-\frac{4S_i^3}{\Delta_i(\alpha)^2}+O(S^4)\,.
\end{aligned}
$$

Here the coefficient $b_{1i}(S)$ is the exact result obtained by comparing the term tending to $\frac{1}{x}$ at $x=\infty$ in (2.5) and (5.13). We would like to remark that the inversion formula (5.3) for ρ_i^2 to order S^2 is sufficient to evaluate the other coefficients $b_{\nu i}(S)$'s to order S^3. Having explicitly known y in the expansion form (5.15) it is easy to evaluate the disc amplitude (2.14) for the n-cut solution. We then see that for $n=2$ it exactly reproduces the disc amplitude given by (4.18). Moreover we note that the evaluation of (2.14) to order S^2 gives the disc amplitude (4.2) in the extended form to the generic n, which was obtained by the $\frac{1}{N}$ expansion in [10].

6. Complex curve for the matrix model and the $\mathcal{N}=1$ supersymmetric QCD

In the literature it is discussed that introducing N_f chiral multiplets as (2.1) induces single pole structure on the Riemann surface Σ given below (2.8). But the formula (2.14) does not reflect any structure of the kind. The n-cut solution (2.7) determining the distribution $\rho(\lambda)$ is merely the leading solution to the saddle point equation(2.5) in the $\frac{1}{M}$ expansion. That is, in solving (2.5) the pole term as well as non-planer contributions were neglected. They could contribute to $F_\chi, \chi\leq 1$. We shall explain that nonetheless the formula (2.14) with (2.7) gives the correct formula for the disc amplitude F_1.

Beyond the leading approximation we have to use the loop equation based on the saddle point equation (2.5) [15]. A clever method [16] is to study the loop equation is to rewrite the saddle point equation (2.5) as

$$
V'(x,g_s)-2\int d\lambda\,\frac{\varrho(\lambda,g_s)}{x-\lambda}=0\,, \tag{6.1}
$$

with

$$V(\lambda, g_s) \equiv V(\lambda) + g_s \sum_{I=1}^{N_f} \log(\lambda + m_I)\,. \tag{6.2}$$

Consequently the partition function (2.6) is rewritten as

$$Z = \exp\biggl(-\frac{1}{g_s^2}\biggl[\int d\lambda\, \rho(\lambda, g_s) V(\lambda, g_s) - \\ - \int\int d\lambda\, d\lambda'\, \rho(\lambda, g_s)\rho(\lambda', g_s) \log(\lambda - \lambda')\biggr]\biggr)\,. \tag{6.3}$$

Using the loop equation we can compute $\rho(\lambda, g_s)$ recursively as

$$\rho(\lambda, g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} \rho_g(\lambda, g_s)\,.$$

It is known that non-planer contributions of (6.3) add up to the effective action of the $\mathcal{N} = 1$ supersymmetric QCD (2.3) as the supergravity effects [2, 17]. Since we are not interested in them in this paper, it suffices to consider the planer contribution $\rho_0(\lambda, g_s)$ alone. Then the n-cut planer solution, which was previously given by (2.7), is now given by a hyperelliptic function with single poles at $x = -m_I$ and the branch cuts $[\lambda_{2i}(g_s), \lambda_{2i+1}(g_s)]$. (See (3.29) in [16] for the explicit form.) That hyperelliptic function determines $\rho_0(\lambda, g_s)$, with which we can compute the partition function (6.3). Expanding the result in series of g_s we find the amplitudes F_2 and F_1 in (2.4). In [16] they were computed for the one-cut planer solution along this line. They found F_1 to be in complete agreement with (3.1). It strongly supports that the calculation of F_1 by (2.14) is correct for the multi-cut solution as well. Thus it suffices to work with the simpler hyperelliptic function (2.7), unless we are interested in the amplitudes $F_\chi, \chi \leq 0$ which modify (2.3) as the supergravity correction.

Then one may ask where the single pole structure due to the introduction of N_f chiral multiplets shows up. It does in the effective action (2.3) of the $\mathcal{N} = 1$ supersymmetric QCD. To explain this we shall look into the effective action (2.3) following [11]. It can be derived with no help of the matrix model. Namely we consider the gauge invariant quantities

$$T(x) = \mathrm{tr}\,\frac{1}{x - \Phi}\,, \qquad R(x) = -\frac{1}{32\pi^2}\,\mathrm{tr}\,\frac{W_\alpha W^\alpha}{x - \Phi}$$

where W_α is the gluino chiral superfield. The effective action is obtained by solving the anomaly equations [18] for these quantities. The solution is again characterized by their analytic structures in the Riemann surface Σ given below (2.8). Namely $R(x)$ has branch cuts along some number of real lines on the complex x-plane, say, $[\lambda_{2i}, \lambda_{2i+1}], i = 1, 2, \ldots, n$, while $T(x)$ has simple poles at $\tilde{P}, P, q_I$ with residues $-N, N - N_f, 1$ respectively. Here $q_I, I = 1, 2, \ldots, N_f$ is the point at $x = -m_I$ on the first sheet. They are given by the following hyperelliptic functions

$$R(x) = y - V'(x)\,, \tag{6.4}$$

$$T(x) = \frac{d}{dx}\log\Bigl[\sqrt{P(x)^2 - \alpha B(x)} - P(x)\Bigr] \equiv \frac{d}{dx}\log\psi(x)\,, \tag{6.5}$$

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with y^2 the polynomial with degree M_0 given by (2.7), $P(x)$ a polynomial with degree N, $B(x) = \prod_I^{N_f}(x + m_I)$ and α a constant. The polynomials y^2, $P(x)$ and $B(x)$ are related by

$$P(x)^2 - \alpha B(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_{2n}) H(x)^2 \,,$$

with $H(x)$ a polynomial of degree $N - n$. This relation guarantees that the function $\psi(x)$ in (6.5) is single valued on Σ. By using (6.4) and (6.5) the effective action (2.3) was written in a geometrical form such that

$$W_{\text{eff}} = \frac{1}{2\pi i} I(T, R) \,, \tag{6.6}$$

with $I(\ ,\ )$ defined by (7.21) in [11]. Thus we can see in the quantity $T(x)$ the pole structure due to the introduction of chiral multiplets. It is worth emphasizing that this geometrical structure of the effective action has nothing to do with the $\frac{1}{N}$ expansion.

7. Conclusions

To the author's knowledge the disc amplitude for the multi-cut solution was calculated only the diagramatic expansion in the literature. The result was merely reported to order S^2 for the generic n-cut solution [10]. The calculation to higher orders can be hardly done, because the diagramatic expansion becomes complicate as the degree of the polynomial scalar potential increases. To confront with this situation we exploited the alternative method, i.e., the complex analysis of the hyperelliptic curve. In section 4 the method was worked out to order S^3 for the disc amplitude for the 2-cut solution. Dealing with the hyperelliptic curve it is relatively easy to extend the calculation to higher orders. In section 5 the argument was generalized for the generic n-cut solution. By using a more systematic algorithm than in section 4, the disc amplitude for the n-cut solution was calculated up to order S^3. There we have given the inversion formula for γ_i to order ρ^4. We would like remark that it was sufficient to give the formula to order ρ^4 for the calculation of the planar amplitude to order S^3 [13]. It was checked that the disc amplitude thus obtained indeed reproduces the one for the 2-cut solution obtained by the $\frac{1}{N}$ expansion in [10]. The algorithm in this paper allows us to generalize the calculation of the disc amplitude for the n-cut solution to higher orders.

Finally we would like to remark that studying the expansion (5.15) of the DV differential ydx to higher orders is important not only for the computation of the amplitudes F_2 and F_1. The important ingredient of the study is the calculation of ρ_i and γ_i as (5.3) and (5.6). It amounts to finding the positions of the branch-points, carrying out the local inversion (2.11):

$$\lambda_r(S_1, \ldots, S_n; \alpha_1, \ldots, \alpha_n) \,, \qquad \text{for } M_0 = n \,, \tag{7.1}$$

in sereis of S's. We now think of perturbing the superpotential (2.1) by a monomial superpotential

$$\Delta V(\Phi) = T \Phi^k \,.$$

This induces the Whitham deformation discussed in [6] and T is called the Whitham flow parameter. Then the branch points changes the positions as

$$\lambda_r(S_1, \dots, S_n; \alpha_1, \dots, \alpha_n) \to \lambda_r(S_1, \dots, S_n; \alpha_1, \dots, \alpha_n, T)\,,$$

with $r = 1, 2, \dots, n$. It happens that some pairs of them coincide or the expansion in S's ceases to be convergent, as T varies with S's fixed. The former case is a geometrical transition reducing genus of the Riemann surface Σ. On the other hand the latter indicates that the local inversion (7.1) gets no longer single valued at some value of T. It might hint at a geometrical transition increasing genus. In order to understand these geometrical transitions by the Whitham deformation it is important to study $\lambda_r(S_1, \dots, S_n; \alpha_1, \dots, \alpha_n, T)$ in sereis of S's and T.

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